GSS Stellarators - Optimization in theory

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For a review on any of the following topics, please see the corresponding sections in the tutorial document (https://arxiv.org/pdf/1908.05360.pdf):

- §4.2 Lagrangian mechanics
- §5.2 Gyroaveraged Lagrangian
- §6.1 Canonical cylindrical coordinates
- §9.3 Boozer coordinates

The following sections of the tutorial discuss the topic of today's lecture, stellarator design and optimization, in more detail:

- §12.1 Quasisymmetry
- §13 Optimization for stellarator design

1. Overview

Objectives for stellarator design:

(1) Large volume of nested flux surfaces ("integrability")
  - Reduce magnetic islands and stochastic layers (Figure 1)

(2) Confinement of single particle trajectories
  - Particles stay close to magnetic surfaces on average

(3) Confinement of particles with collisions ("neoclassical transport")
  - Reduced collisional heat and particle diffusion

(4) Stability to perturbations
  - MHD stability (fast time scale, long wavelength)
  - Microstability (slower time scale, small wavelength)

(5) Practical construction
  - Sufficiently simple coils
  - Distance between coils
  - Distance between coils and plasma

In this lecture, we will focus on just a few aspects of stellarator design. In particular, we will discuss quasisymmetry from a theoretical and pratical point of view.

2. Quasisymmetry

2.1. Lagrangian mechanics reminder. We begin with the Lagrangian for the single-particle dynamics in a general magnetic field,

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{m|\dot{\mathbf{r}}|^2}{2} + q\mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}}, \tag{1}$$

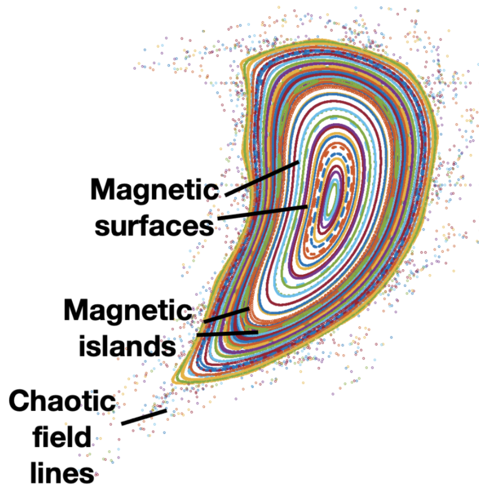


FIGURE 1. Poincarè section of the magnetic field evaluated from the NCSX modular coils. Here a magnetic field is followed many times around a device until it intersects a plane at constant toroidal angle. Colors differentiate different magnetic field lines.

where $\mathbf{B} = \nabla \times \mathbf{A}$. Here we have considered the limit of stationary fields and neglected any electric fields. (Electric fields can be accounted for in all of this discussion, but will only complicate the expressions.) Recall that the particle trajectories, $\mathbf{r}(t)$, are obtained from the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L(\mathbf{r},\dot{\mathbf{r}})}{\partial \dot{\mathbf{r}}}\right) = \frac{\partial L(\mathbf{r},\dot{\mathbf{r}})}{\partial \mathbf{r}}. \tag{2}$$

We remark that if the Lagrangian has an ignorable coordinate, $\partial L/\partial r_i = 0$, then this yields a conserved momentum, $p_i \equiv \partial L/\partial \dot{r}_i$,

$$\rightarrow \frac{dp_i}{dt} = 0. \tag{3}$$

2.2. Confinement in axisymmetry. The existence of such a conserved quantity has an important impact on confinement. To explore this, we consider the limit of axisymmetry. Axisymmetry is most apparent if we employ standard cylindrical coordinates (Figure 2),

$$\mathbf{r} = R\hat{\mathbf{R}} + Z\hat{\mathbf{z}} \tag{4}$$

$$\dot{\mathbf{r}} = \dot{R}\hat{\mathbf{R}} + R\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{Z}\hat{\mathbf{z}} \tag{5}$$

to express the Lagrangian as,

$$L(R, Z, \phi, \dot{R}, \dot{Z}, \dot{\phi}) = \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{Z}^2\right) + q\left(A_R\dot{R} + A_\phi R\dot{\phi} + A_Z\dot{Z}\right). \tag{6}$$

If we consider an axisymmetric magnetic field,

$$\frac{\partial A_R}{\partial \phi} = \frac{\partial A_\phi}{\partial \phi} = \frac{\partial A_Z}{\partial \phi} = 0, \tag{7}$$

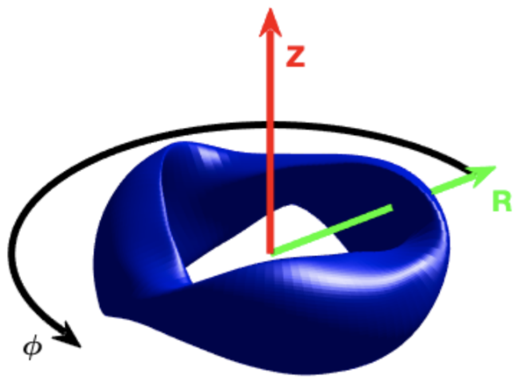


FIGURE 2. Cylindrical coordinate system.

this implies that $\partial L/\partial\phi = 0$, and yields a corresponding conserved momentum,

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi} + qRA_\phi. \tag{8}$$

In the limit of large field strength, we can note that the first term is much larger than the second term. We compute the approximate scaling of their ratio, using an approximate length scale L, approximate field strength B, and thermal velocity v_t,

$$\frac{mR^2\dot{\phi}}{qRA_\phi} \sim \frac{mv_tL}{qL^2B} \sim \frac{v_t}{\Omega L}. \tag{9}$$

Here we have noticed the appearance of the gyrofrequency, $\Omega \equiv qB/m$. For a strongly magnetized plasma, this ratio $\rho_* \equiv v_t/(\Omega L) \ll 1$ (typically, $\sim 1/100$ in magnetic confinement devices). This implies that, to a good approximation, $p_\phi \approx qRA_\phi$ is conserved. In axisymmetic magnetic fields, it turns out that RA_ϕ is a flux function. Thus this conserved momentum tells us, to a good approximation, axisymmetry implies particles stay close to flux surfaces.

2.3. Guiding center Lagrangian. We are interested in more general symmetries that enable departure from axisymmetry. We will specifically look for symmetries of the *guiding center motion*. Recall that the position vector of a particle, $\mathbf{r}$, can be decomposed into the guiding center, $\mathbf{R}$, and gyroradius, $\boldsymbol{\rho}$,

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}, \tag{10}$$

where $\boldsymbol{\rho}$ accounts for the fast gyration of the field and $\mathbf{R}$ accounts for the averaged motion (Figure 3).

In the limit of small gyroradius (large gyrofrequency) in comparison with typical scales of interest, it can be costly to evaluate the full dynamics from the single-particle Lagrangian. Instead, we can perform an asymptotic expansion with respect to the small parameter ρ_* to obtain a Lagrangian for the guiding center, $\mathbf{R}$,

$$\mathcal{L}(\mathbf{R}, \dot{\mathbf{R}}) = \frac{m}{2}\left(\dot{\mathbf{R}} \cdot \hat{\mathbf{b}}(\mathbf{R})\right)^2 + q\mathbf{A}(\mathbf{R}) \cdot \dot{\mathbf{R}} - \mu B(\mathbf{R}), \tag{11}$$

where $\mu = mv_\perp^2/(2B)$, the magnetic moment, is a conserved quantity along a trajectory, and $\hat{\mathbf{b}} = \mathbf{B}/B$ is the unit vector in the direction of the magnetic field. For the purposes of this discussion, it is not

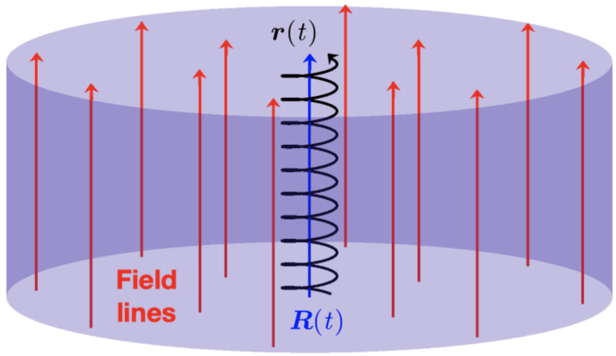


FIGURE 3. The guiding center trajectory (blue) describes the particle's motion (black) after averaging over the fast gyration.

necessary to understand how this Lagrangian is obtained, but an interested reader can refer to the relevant section (§5.2) in the tutorial for details. We seek symmetries of the guiding center Lagrangian in order to obtain conserved momenta along trajectories.

We will find that this analysis will be most clear by applying Boozer coordinates to parameterize the guiding center position, $\mathbf{R}(\psi, \vartheta_B, \varphi_B)$. From the chain rule, we obtain the velocity vector as,

$$\dot{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial \psi}\dot{\psi} + \frac{\partial \mathbf{R}}{\partial \vartheta_B}\dot{\vartheta}_B + \frac{\partial \mathbf{R}}{\partial \varphi_B}\dot{\varphi}_B. \tag{12}$$

Recall that the magnetic field can be written in the covariant form as,

$$\mathbf{B}(\psi, \vartheta_B, \varphi_B) = G(\psi)\nabla\varphi_B + I(\psi)\nabla\vartheta_B + K(\psi, \vartheta_B, \varphi_B)\nabla\psi, \tag{13}$$

and the contravariant form as,

$$\mathbf{B}(\psi, \vartheta_B, \varphi_B) = \nabla\psi \times \nabla\vartheta_B - \iota(\psi)\nabla\psi \times \nabla\varphi_B. \tag{14}$$

To be consistent with the contravariant form, we can choose the vector potential $\mathbf{A}$ as,

$$\mathbf{A}(\psi, \vartheta_B, \varphi_B) = \psi\nabla\vartheta_B - \psi_P(\psi)\nabla\varphi_B, \tag{15}$$

where $\psi_P(\psi)$ is the poloidal flux, satisfying $\psi_P'(\psi) = \iota(\psi)$. This corresponds to one choice of the gauge. Recall that the gradient of any function, ∇f, can be added to $\mathbf{A}$ without changing the magnetic field. The resulting trajectories are independent of the choice of gauge. Expressing the Lagrangian in this coordinate system, we obtain

$$\begin{aligned}\mathcal{L}(\psi, \vartheta_B, \varphi_B, \dot{\psi}, \dot{\vartheta}_B, \dot{\varphi}_B) = \frac{m}{2B(\psi, \vartheta_B, \varphi_B)^2}\left(\dot{\psi}K(\psi, \vartheta_B, \varphi_B) + \dot{\vartheta}_B I + \dot{\varphi}_B G\right)^2 + q\left(\psi\dot{\vartheta}_B - \psi_P(\psi)\dot{\varphi}_B\right) \\ - \mu B(\psi, \vartheta_B, \varphi_B).\end{aligned} \tag{16}$$

We now seek a symmetry of $\mathcal{L}$ with respect to a particular coordinate. First, we note that the angular dependence of the Lagrangian only enters through the radial covariant component, $K(\psi, \vartheta_B, \varphi_B)$, and the field strength, $B(\psi, \vartheta_B, \varphi_B)$. Recall that the radial covariant component satisfies a magnetic differential equation,

$$\iota(\psi)\frac{\partial K}{\partial \vartheta_B} + \frac{\partial K}{\partial \varphi_B} = \frac{G + \iota(\psi) I}{B^2}\mu_0 p'(\psi) + G'(\psi) + \iota(\psi) I'(\psi), \tag{17}$$

which is driven by the angular dependence of B. This implies that if B possesses a particular symmetry, then K will inherit this symmetry.

In particular, suppose that B depends *only* on a particular combination of the Boozer angles, $\chi \equiv \vartheta_B - N/M\varphi_B$ (here we assume $M \neq 0$). To evaluate this symmetry, we use the coordinates (ψ, θ_B, χ) in our Lagrangian. This implies the existence of a symmetry direction of B and thus $\mathcal{L}$,

$$\frac{\partial B(\psi, \vartheta_B, \chi)}{\partial \vartheta_B} = 0 \to \frac{\partial \mathcal{L}(\psi, \vartheta_B, \chi, \dot{\psi}, \dot{\vartheta}_B, \dot{\chi})}{\partial \vartheta_B} = 0. \tag{18}$$

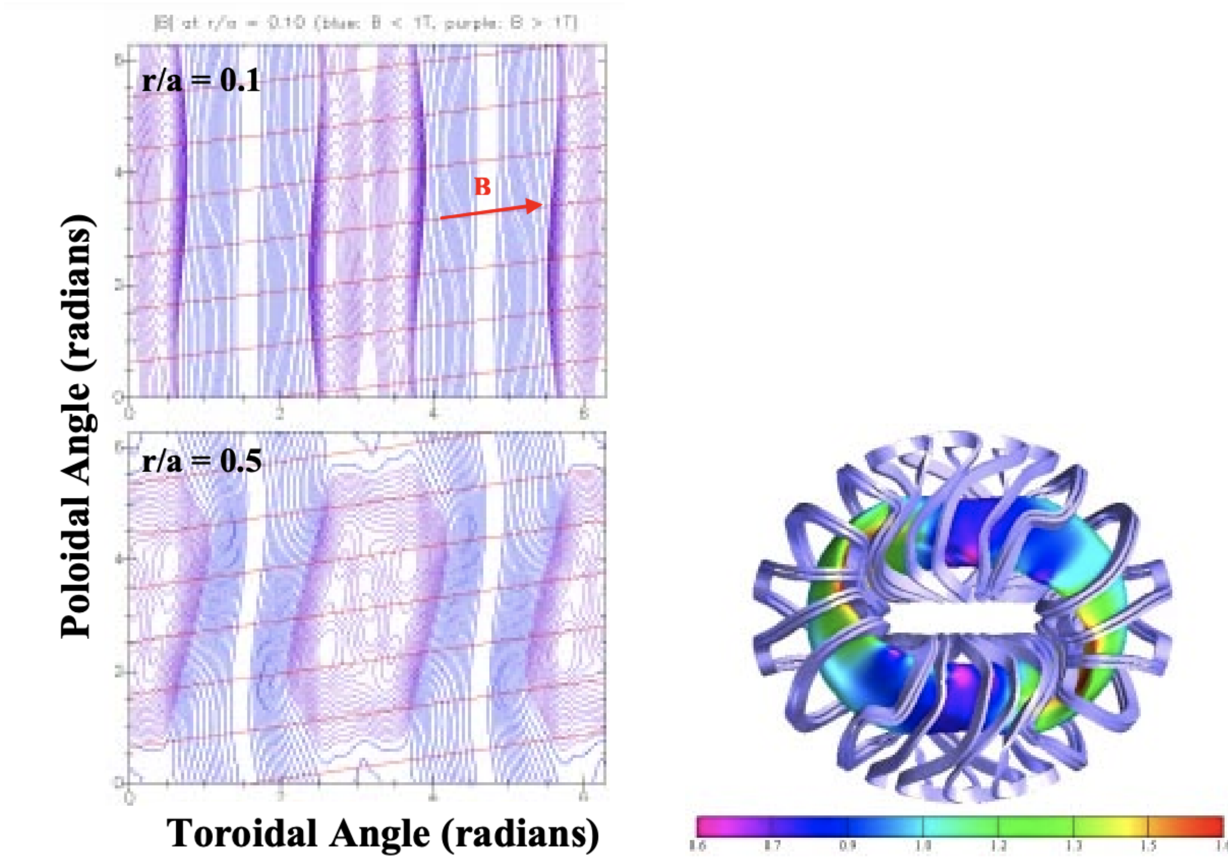
From the Euler-Lagrange equation, we then obtain the following conserved momentum,

$$p_{\vartheta_B} \equiv \frac{\partial \mathcal{L}(\psi, \vartheta_B, \chi, \dot{\psi}, \dot{\vartheta}_B, \dot{\chi})}{\partial \dot{\vartheta}_B} = \frac{m\dot{\mathbf{R}} \cdot \hat{\mathbf{b}}}{B}\left(I + GM/N\right) + q\left(\psi - M/N\psi_P(\psi)\right). \tag{19}$$

In analogy with the analysis of the axisymmetric case, we note that the first term is smaller than the second term by a factor of ρ_*. Thus this symmetry, again, implies that guiding center trajectories stay close to flux surfaces. This symmetry is known as *quasisymmetry*. Unlike axisymmetry, it does not require a continuous symmetry of the magnetic field vector, but only a symmetry of the magnetic field strength on a surface (in Boozer coordinates).

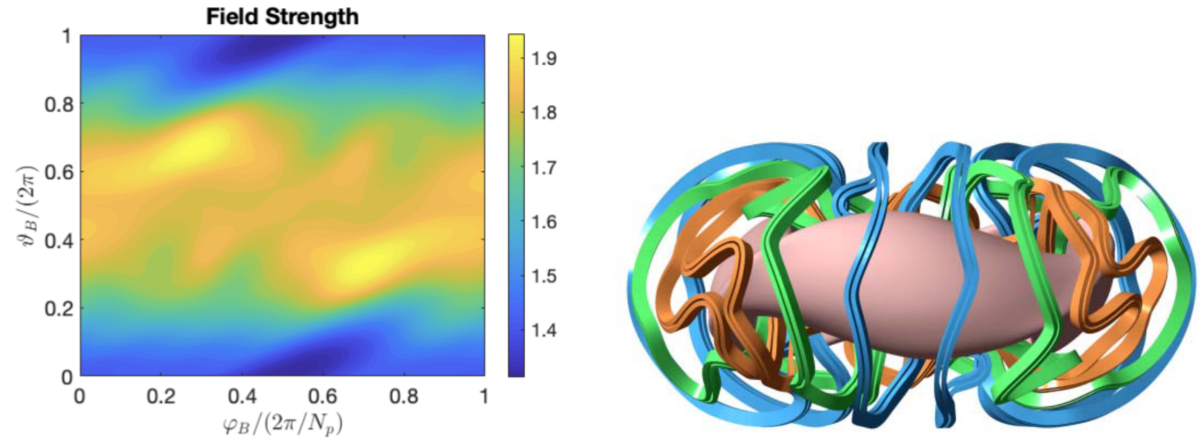
Types of quasisymmetry:

(1) $M = 0$: quasi-poloidal symmetry (Figure 4)
(2) $N = 0$: quasi-axisymmetry (Figure 5)
(3) $M \neq 0$, $N \neq 0$: quasi-helical symmetry (Figure 6)



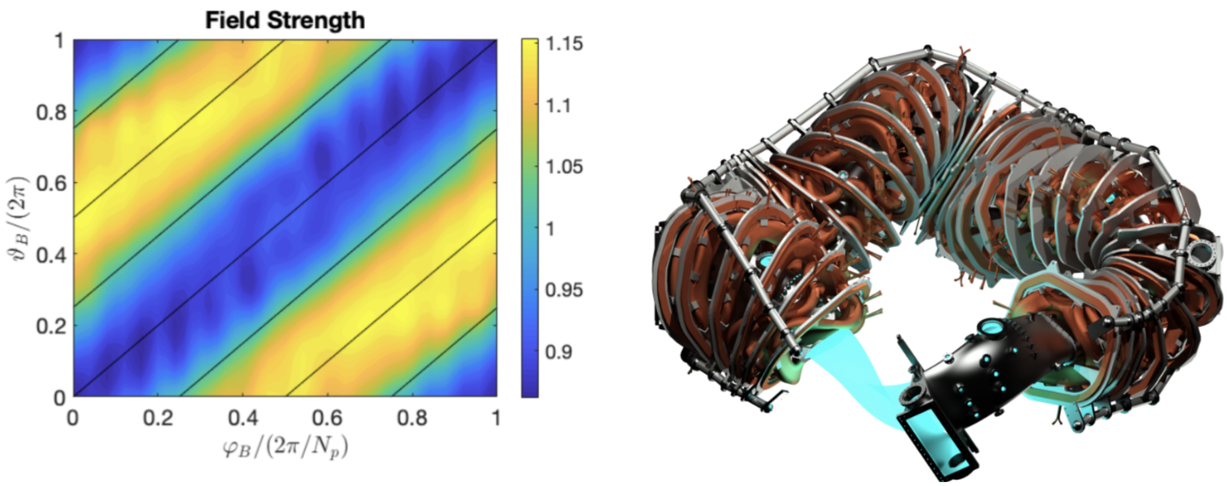
(A) (B)

FIGURE 4. The Quasi-Poloidal Stellarator (QPS) (a) field strength and (b) coils with plasma boundary. [J. F. Lyon et al, *IAEA* 2005]



(A) (B)

FIGURE 5. The National Compact Stellarator eXperiment (NCSX) (a) field strength and (b) coils with plasma boundary. [D. J. Strickler et al, *IAEA* 2006]



(A) (B)

FIGURE 6. The Helically Symmetric eXperiment (HSX) (a) field strength and (b) coils with plasma boundary. [F. S. B. Anderson et al, *Fusion Technology* 1995]